Emerging Photonic Principles and Negative Effective Mass

A Mathematical Treatise in the Tradition of Grothendieck

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Abstract

The sun has a twin sister star, and there is much related to gravity we do not understand today. However, deeper questions related to the standard model remain. Questions related to dark matter, and mass. We will focus in this work on mass. We suspect there are discoveries to be made related to negative effective mass (not negative mass). We present mathematical formulation for mirror symmetry in photonic contexts, SUSY constructions, negative effective mass dynamics, Madelung-fluid formulation, Schrödinger operators on surfaces, Bollobás–Riordan polynomial in network topology, and dark-photon vortex solutions. We hope in this regard we contribute to the study of light and negative effective mass.

1 Mirror Symmetry

Let X be a Calabi–Yau manifold of complex dimension n and Y its mirror. We summarize key identities and differential relations used in enumerative geometry and their photonic analogues.

Definition 1.1. *(Period vector) For a family $\mathcal{X}_z$ with holomorphic n-form $\Omega(z)$, define periods*

$$\Pi_i(z) = \int_{\gamma_i} \Omega(z), \qquad \gamma_i \in H_n(X, \mathbb{Z}).$$

The variation of Hodge structure (VHS) produces Picard–Fuchs operators. Let $\theta = z\frac{d}{dz}$. Example (quintic):

$$\mathcal{L} = \theta^4 - 5z(5\theta + 1)(5\theta + 2)(5\theta + 3)(5\theta + 4).$$

Periods satisfy $\mathcal{L}\Pi(z) = 0$. Expand around $z = 0$:

$$\Pi_0(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_k = \frac{(5k)!}{(k!)^5}.$$

Definition 1.2. *(Yukawa coupling) For moduli coordinate t, define*

$$C_{ttt}(t) = \int_X \Omega \wedge \frac{\partial^3}{\partial t^3}\Omega.$$

Gromov–Witten generating function for genus g:

$$F_g^X(t) = \sum_{d \in H_2(X,\mathbb{Z})} N_{g,d} e^{d \cdot t},$$

with mirror map $q = e^{t(z)}$ obtained from period ratios.

Theorem 1.3 (Kontsevich–Manin (schematic))**.** *Genus-zero Gromov–Witten invariants are encoded by solutions to Picard–Fuchs equations on the mirror family; Yukawa couplings compute 3-point correlators.*

Categorical statement (Homological Mirror Symmetry, HMS):

$$D^b\mathrm{Coh}(X) \simeq \mathrm{Fuk}(Y),$$

with equivalence of triangulated categories; under this equivalence, structure sheaves map to Lagrangian branes and Ext-groups map to Floer cohomology:

$$\mathrm{Ext}^k(\mathcal{E}, \mathcal{F}) \cong HF^k(L_{\mathcal{E}}, L_{\mathcal{F}}).$$

Photonic application : treat a photonic lattice family as a family of complex structures; localized defect modes correspond to objects of $D^b\mathrm{Coh}$ and scattering matrices correspond to morphisms. Note that PF operators and period expansions can be used in numerical matching.

2 SUSY (Supersymmetric Quantum Mechanics and Isospectral Constructions)

2.1 SUSY algebra and factorization

Let H be a Schrödinger operator on $L^2(\mathbb{R})$:

$$H = -\frac{d^2}{dx^2} + V(x), \qquad V : \mathbb{R} \to \mathbb{R}.$$

Assume a superpotential $W(x)$ with

$$A = \frac{d}{dx} + W(x), \quad A^\dagger = -\frac{d}{dx} + W(x),$$

and define

$$H_- = A^\dagger A = -\frac{d^2}{dx^2} + V_-(x), \quad H_+ = AA^\dagger = -\frac{d^2}{dx^2} + V_+(x),$$

with

$$V_\mp(x) = W(x)^2 \mp W'(x).$$

Spectra satisfy $\mathrm{Spec}(H_+) = \mathrm{Spec}(H_-) \setminus \{E_0\}$ under non-degeneracy.

2.2 Index and Witten index

Define supercharges

$$Q = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \quad Q^\dagger = \begin{pmatrix} 0 & A^\dagger \\ 0 & 0 \end{pmatrix}, \quad \{Q, Q^\dagger\} = \mathcal{H} = \begin{pmatrix} H_+ & 0 \\ 0 & H_- \end{pmatrix}.$$

Witten index:

$$\Delta = \mathrm{Tr}\,(-1)^F e^{-\beta\mathcal{H}} = n_B - n_F,$$

invariant under continuous deformations of W.

2.3 Constructive algorithms

Given measured spectrum $\{E_n\}$, construct approximate superpotential W via inverse scattering or Gel'fand–Levitan–Marchenko equations. If $R(k)$ is reflection coefficient, solution kernel $K(x, y)$ satisfies

$$K(x, y) + F(x + y) + \int_x^\infty K(x, s)F(s + y)\, ds = 0,$$

with F computed from scattering data; then

$$W(x) = -2\frac{d}{dx}K(x, x).$$

3 Negative Effective Mass

Our insights now lead into a new angle on negative effective mass. Not to be confused with negative mass. Let $E(k)$ be the band dispersion. Define

$$\frac{1}{m^*(k)} = \frac{1}{\hbar^2}\frac{d^2E}{dk^2}.$$

Local expansion near extremum k_0:

$$E(k) = E(k_0) + \frac{\hbar^2}{2m^*}(k-k_0)^2 + O((k-k_0)^3).$$

If $m^* < 0$, standard dynamics invert.

3.1 Field-theoretic Lagrangian and linear stability

Field $\psi(t, \mathbf{x})$ with effective mass m^* and nonlinearity g:

$$\mathcal{L} = i\hbar\psi^*\partial_t\psi - \frac{\hbar^2}{2m^*}|\nabla\psi|^2 - \frac{g}{2}|\psi|^4 - V(\mathbf{x})|\psi|^2.$$

Euler–Lagrange yields nonlinear Schrödinger (NLS):

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m^*}\Delta\psi + g|\psi|^2\psi + V\psi.$$

Plane-wave background $\psi_0 e^{-i\mu t/\hbar}$, perturbation $\delta\psi \sim e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ gives Bogoliubov spectrum:

$$\hbar\omega(\mathbf{k}) = \sqrt{\epsilon_k\,(\epsilon_k + 2g|\psi_0|^2)}, \quad \epsilon_k = \frac{\hbar^2k^2}{2m^*}.$$

When $m^* < 0$, ϵ_k changes sign leading to bands with imaginary ω, indicating modulational instability under parameter regimes where $\epsilon_k(\epsilon_k + 2g|\psi_0|^2) < 0$.

3.2 Soliton existence and Vakhitov–Kolokolov (VK) criterion

Stationary ansatz $\psi(\mathbf{x}, t) = \phi(\mathbf{x})e^{-i\mu t/\hbar}$ solves

$$-\frac{\hbar^2}{2m^*}\Delta\phi + g|\phi|^2\phi = (\mu - V)\phi.$$

For 1D localized solutions, define power $P(\mu) = \int |\phi|^2 dx$. VK stability condition:

$$\frac{dP}{d\mu} > 0 \quad \text{(stable)}, \qquad \frac{dP}{d\mu} < 0 \quad \text{(unstable)}.$$

Sign of m^* influences existence domain of bright/dark solitons. CERN validations are needed to confirm the fysical presence of these states.

4 Fluid Theory: Madelung Transform and Hydrodynamics

Madelung substitution:

$$\psi = \sqrt{\rho}\,e^{iS/\hbar}, \quad v = \frac{1}{m^*}\nabla S.$$

Continuity and Euler-type equations:

$$\partial_t\rho + \nabla\cdot(\rho v) = 0,$$

$$m^*(\partial_t v + (v \cdot \nabla)v) = -\nabla V - \nabla Q - g\nabla\rho,$$

with quantum potential

$$Q = -\frac{\hbar^2}{2m^*}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}.$$

Note: sign dependence on m^* throughout. Linearize about $\rho = \rho_0, v = 0$:

$$\partial_t^2 \delta\rho = c_s^2 \Delta\delta\rho - \frac{\hbar^2}{4m^{*2}}\Delta^2\delta\rho, \quad c_s^2 = \frac{g\rho_0}{m^*}.$$

If $m^* < 0$, $c_s^2 < 0$ and dispersion yields exponential growth for long wavelengths.

4.1 Vortex quantization

Circulation:

$$\Gamma = \oint v \cdot d\ell = \frac{2\pi\hbar n}{m^*}, \qquad n \in \mathbb{Z}.$$

Sign of m^* reverses orientation convention for vorticity. It seems extremely likely, there is some kind of negative effective mass behaviour present in nature.

5 Schrödinger's Particles on Surfaces

Let (M, g) be a smooth 2D Riemannian manifold (surface) isometrically embedded in $\mathbb{R}^3$. Consider constraining potential $V_\perp(\xi)$ normal to surface; in the thin-layer limit (da Costa) effective Hamiltonian on surface reads:

$$H = -\frac{\hbar^2}{2m}\Delta_g + V_g + V_\parallel,$$

where

$$V_g = -\frac{\hbar^2}{2m}\left(H^2 - K\right),$$

with H mean curvature and K Gaussian curvature. This leads us toward covering compact surfaces.

5.1 Spectral asymptotics

Weyl law for compact surface M:

$$N(E) = \#\{\lambda_j \le E\} = \frac{\mathrm{Area}(M)}{4\pi}E - \frac{\sqrt{E}}{4\pi}\int_{\partial M}\kappa\, ds + o(\sqrt{E}),$$

with curvature corrections entering at lower order through V_g. For large E eigenvalue counting approaches the flat-space leading term with curvature-dependent corrections.

5.2 Let us expand our idea: examples

Sphere $S^2(R)$: $H = 1/R$, $K = 1/R^2$, thus $V_g = -\frac{\hbar^2}{2m}(1/R^2 - 1/R^2) = 0$; Laplacian eigenvalues well-known $\lambda_\ell = \ell(\ell+1)/R^2$. Torus and higher genus surfaces: curvature induced localization and band splitting.

6 The Bollobás–Riordan Polynomial and Network Topology

Definition 6.1. *These math can help visualize the idea. A ribbon graph $G = (V, E)$ is a graph with a cyclic order of half-edges at each vertex; it corresponds topologically to an embedding of G in a surface Σ. The Bollobás–Riordan polynomial $R_G(x, y, z)$ is defined by the state-sum*

$$R_G(x, y, z) = \sum_{H \subseteq E} (x-1)^{r(G)-r(H)}(y-1)^{n(H)} z^{k(H)-bc(H)+n(H)},$$

where $r(H) = |V| - k(H)$, $n(H) = |H| - r(H)$, $bc(H)$ is number of boundary components of the cellular embedding of subgraph (V, H).

6.1 Contraction-deletion relations

If e is an ordinary edge (not a loop),

$$R_G = R_{G\setminus e} + R_{G/e}.$$

For twisted loops relations modify powers of z accordingly.

6.2 Relation to Potts-type partition functions

It must be possible to define generalized partition function on ribbon graph (Potts on surface) with genus weight; then coefficients of R_G correspond to weighted counts of coloring/configurations. The Photonic side one could interpret resonator networks as ribbon graphs; spectral properties correlate with polynomial coefficients.

7 Dark Photon Vortex Formation

Dark photons in turn are abstract, however one could consider Abelian gauge fields A_μ (visible) and X_μ (hidden/dark) with kinetic mixing χ. Matter complex scalar ϕ charged under hidden $U(1)_X$ with covariant derivative $D_\mu = \partial_\mu + ig_X X_\mu$.

Lagrangian (2+1 or 3+1 dimensional as appropriate):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}X_{\mu\nu}X^{\mu\nu} - \frac{\chi}{2}F_{\mu\nu}X^{\mu\nu} + |D_\mu\phi|^2 - V(\phi),$$

where

$$V(\phi) = \frac{\lambda}{4}(|\phi|^2 - v^2)^2.$$

Spontaneous symmetry breaking yields $m_X = g_X v$, and vortex ansatz (cylindrical symmetry):

$$\phi(r, \theta) = f(r)e^{in\theta}, \quad X_\theta(r) = \frac{n}{g_X r}a(r), \quad X_r = 0.$$

Equations reduce to radial ODEs:

$$f'' + \frac{1}{r}f' - \frac{n^2}{r^2}(1-a)^2 f - \lambda(f^2 - v^2)f = 0,$$

$$a'' - \frac{1}{r}a' - 2g_X^2 f^2(1-a) = 0,$$

(modulo normalization constants and mixing contributions). Boundary conditions:

$$f(0) = 0,\ f(\infty) = v; \quad a(0) = 0,\ a(\infty) = 1.$$

7.1 Bogomol'nyi completion

It is interesting to note the importance of symmetry breaking. Let us now look at vortex solutions. At critical coupling $\lambda = 2g_X^2$ (in appropriate units), energy per unit length E admits Bogomol'nyi bound and BPS equations:

$$D_1\phi \pm iD_2\phi = 0, \quad B_X = \pm(g_X)(|\phi|^2 - v^2),$$

with $B_X = (\nabla \times X)_z$ and integer flux quantization

$$\Phi_X = \int B_X \, d^2x = \frac{2\pi n}{g_X}.$$

Thus vortex solutions classified by $n \in \mathbb{Z}$; energy $E \geq 2\pi v^2|n|$.

8 Concluding remarks on negative effective mass

It is likely some sort of negative effective mass, has certain influence in nature, and these might require further study. Our math is inconclusive, yet strongly points towards the existence of these behaviours.

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